# Covert Discrimination and Self-promotion

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October 18, 2022

**Abstract**

Agents with similar skill may differ in their ability to self-promote. We consider the problem of a manager who uses an efficient, anonymous contest to extract effort from equally productive workers who differ in their ability to win the contest. If the prize is fixed, it is often possible to discriminate against the stronger player despite anonymity. However, full surplus extraction is not typically possible. If the designer can endogenize the prize, full surplus extraction is possible in an all-pay auction as long as a single-crossing condition is met. In the optimal contest, the worker with the better self-promotion technology is endogenously offered a lower expected prize. Because the contest is anonymous, this discrimination is *covert*.

**Keywords:** contest design, auctions, spillovers, discrimination, anonymity.
**JEL Codes:** C72, D44, D62, D74, D82

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# 1 Introduction

Many economic settings are described by competition between individuals for a prize. However, these competitions can be asymmetric in the sense that contestants may differ in their inherent ability to win the contest. This heterogeneity can reduce competitiveness and decrease effort. When this happens, a contest designer may want to level the playing field by discriminating in favor of the weaker player. This approach has been explored by the large empirical and theoretical literature on discrimination as a way of benefiting weaker players (Holzer and Neumark, 2000; Roemer, 1998) and maximizing aggregate effort (Brown, 2011; Franke et al., 2013).

These advantages in contests are not necessarily advantages in productive output. In employment contexts, workers sometimes win promotions by appearing busier or appearing more productive than competitors rather than actually being more productive. For example, workers can stay in the office later than their bosses, fill up their calendars, or do a better job of internally selling their work output. Because this self-promotion is internal, it does not further the goals of the organization.

We consider one such problem where a manager designs a contest for a promotion or bonuses to encourage equally productive workers to exert effort. While the workers are equally capable of producing valuable work output, the workers differ in their ability to self-promote. That is, the output that the manager actually observes may be "enhanced" by self-promotion.

If the manager knows who is self-promoting and can discriminate directly against workers, this self-promotion can be corrected for. This is the solution considered in much of the contest design literature. Typically, discrimination is *overt* – that is, the designer makes the contest depend on the identities of the participants. For example, Mealem and Nitzan, 2014 studies optimal reward schemes in a complete information all-pay auction (APA) where the stronger player is offered a smaller reward for winning than her opponent. These methods require both that the designer knows the identities of the contestants and is allowed to design a contest that treats players asymmetrically. In practice, it may not be viable – or even desirable – to construct a contest that overtly favors one player. This could be due to concerns about fairness. For example, the manager may be unable to offer larger bonuses to apparently less productive employees. It could also be due to information. For example, the designer may not be able to identify the weaker player.

We instead consider discrimination that is *covert* in the sense that neither the value nor the allocation of the prize depends on the identities of the players. Indeed, the only asymmetries are the players' inherent differences in their self-promotion abilities. Such schemes have the advantage of apparent fairness, since players are treated symmetrically. In order to achieve discrimination in this setting, the key is to make sure stronger players are still disadvantaged *in equilibrium*.

We show that a manager who must award a promotion and is not allowed to adjust the value of the prize may be able to achieve full surplus extraction, but often is unable to do so. To achieve this extraction, the designer must grant a larger expected prize to the weaker player. No commonly used contest has this property. We introduce a contest that does, an all-pay auction combined with a difference-form contest, and show that it can achieve full surplus extraction in some settings. This requires a single-crossing type condition in the self-promotion capacities, but the property is not sufficient.

When the manager is allowed to adjust the value of the prize endogenously (e.g., a bonus) the single-crossing property is sufficient to ensure full surplus extraction. The optimal contest is an all-pay auction with spillovers and no bid caps as in (Betto and Thomas, 2021). This contest allows the manager to completely level the playing field – effectively making the restriction of anonymity useless. This game has a unique Nash equilibrium. So, the outcome is uniquely implemented.

This paper contributes to the literature on optimal design of contests (eg. Che and Gale, 2003; Mealem and Nitzan, 2014) by allowing the designer to introduce spillovers to contests. We especially relate to Jönsson and Schmutzler, 2013, which studies the design of all-pay auctions with endogenous prizes. Unlike their paper, we study contests with spillovers where the prize depends on the opponents' effort rather than the players' own. This difference is crucial and drives our results.

We also contribute to the literature on optimal covert discrimination in Tullock contests (eg. Epstein et al., 2013; Nti, 1999), which studies how the contest success function of the Tullock contest can be adjusted (symmetrically) to maximize revenue. In contrast, we allow for the designer to pick any symmetric contest success function, and adjust the prize (symmetrically) to maximize revenue. This freedom allows for a more literal sort of discrimination where the weaker player receives a larger prize in equilibrium – something that does not happen in any commonly studied contest framework.

# 2 Model

One manager designs a contest to provide an effort incentive to two workers. Each worker, $i$, selects costly output $y_i \in \mathbb{R}_+$ and transforms it using their strictly increasing self-promotion technology, $f_i : \mathbb{R}_+ \to \mathbb{R}_+$.

The manager cannot directly observe $y_i$. The manager instead observes the transformed output, which we denote the worker's score: $s_i \equiv f_i(y_i)$. The manager wants to maximize the sum of real outputs (not scores). That is, the manager's objective is to maximize $E[y_1 + y_2]$. Note that observing the sum of actual outputs does not allow the manager to back out the individual outputs and does not allow for direct discrimination.

The manager has complete *anonymous* information. That is, the manager knows $f_1$ and $f_2$, but does not know which corresponds to each employee. Alternatively, the manager does know which worker is which, but is unable to discriminate directly.

The contest that the manager designs consists of two components: (1) a prize, $v_i(s_i; s_{-i})$, which may depend on the winner and loser's score and (2) a contest success function, $p_i(s_i; s_{-i})$, which defines each player's probability of winning the prize given the profile of scores.

The payoff of a worker who plays $s_i$ against an opponent who plays $s_{-i}$ receives the expected prize less the output provided to the manager:

$$U_i(s_i, y_i; s_{-i}) \equiv p_i(s_i; s_{-i})v_i(s_i; s_{-i}) - y_i.$$

## 2.1 Assumptions

We assume that $f_1, f_2$ are strictly increasing, continuously differentiable, and ranked. So, $f_1(x) > f_2(x)$ for all $x > 0$ without loss. We make the following restrictions on the manager:

**Assumption 1** (Budget constraint)**.** *There is a limited amount of expected prize (indexed to 1):*

$$EV \equiv E[p(s_1; s_2)v(s_1; s_2) + p(s_2; s_1)v(s_2; s_1)] \leq 1$$

**Assumption 2** (Efficiency)**.** *There is one prize and it is always allocated. That is,*

$$p_1(x;y) + p_2(y;x) = 1 \quad \textit{and} \quad v_i(x;y) > 0 \quad \forall i, x, y.$$

**Assumption 3** (Anonymity)**.** *The probability of winning the prize and the prize itself do not depend on the identity of the payer. That is,*

$$p_1(x;y) = p_2(x;y) \quad \textit{and} \quad v_1(x;y) = v_2(x;y) \quad \forall x, y.$$

As a result of anonymity, we can remove all subscripts from $p$ and $v$. The assumption of efficiency is very important. Without it, the optimal contest would be an auction with a reserve bid set at $f_1(1)$ such that only Worker 1 participated.

# 3 Results

Because the self-promotion technology, $f_i$, is strictly increasing, it is invertible. We can use this to rewrite each worker's payoff entirely in terms of the score:

$$U_i(s_i; s_{-i}) \equiv p(s_i; s_{-i})v(s_i; s_{-i}) - f_i^{-1}(s_i).$$

Whatever the equilibrium of the contest, the following condition must hold for all $x$ in the support of Player $i$'s strategy.

$$\bar{u}_i \equiv E[p(x; s_{-i})v(x; s_{-i})] - f_i^{-1}(s_i) \tag{1}$$

where $\bar{u}_i$ is the payoff of Player $i$ and $s_{-i}$ is a (possibly deterministic) random variable representing the equilibrium strategy of Player $i$'s opponent. This implies that the principal's expected revenue can be rewritten as follows:

$$\begin{aligned} E[y_1 + y_2] &= E[f_1^{-1}(s_1) + f_2^{-1}(s_2)] \\ &= E[p(s_1; s_2)v(s_1; s_2) + p(s_2; s_1)v(s_2; s_1)] - \bar{u}_1 - \bar{u}_2 \\ &= EV - \bar{u}_1 - \bar{u}_2. \end{aligned}$$

So, if the expected value of the contest is held constant, the principal's problem is equivalent to minimizing the payoffs of the workers. When we say that a contest extracts the full surplus, we mean that $EV = 1$ and $\bar{u}_1 + \bar{u}_2 = 0$.

## 3.1 Direct discrimination

If the manager could discriminate directly, it is easy to extract the full surplus in an asymmetric all-pay auction with $v(s_i; s_{-i}) \equiv 1$ and

$$p_i(s_i; s_{-i}) \equiv \begin{cases} 0 & \text{if } f_i^{-1}(s_i) < f_{-i}^{-1}(s_{-i}) \\ \frac{1}{2} & \text{if } f_i^{-1}(s_i) = f_{-i}^{-1}(s_{-i}) \\ 1 & \text{if } f_i^{-1}(s_i) > f_{-i}^{-1}(s_{-i}). \end{cases}$$

This all-pay auction is known to extract the full surplus of all participants because it is equivalent to a symmetric all-pay auction over real output, $y$.

## 3.2 Fixed prizes

We enforce Assumptions 1-3 and also require that $v(s_i; s_{-i}) \equiv 1$. In this case, the optimal contest does not extract the full surplus. It is an all-pay auction with bid caps.

**Lemma 1.** *Full surplus extraction is possible only if Worker 2 wins the prize with probability greater than one half in equilibrium.*

*Proof.* By anonymity, efficiency, and the fixed prize assumption, either player can obtain an expected prize of 0.5 by copying the opponent's strategy. As a result,

$$\begin{aligned} \frac{1}{2} - E[f_1^{-1}(s_2)] &\leq E[U_1(s_1; s_2)] \\ &\leq E[U_1(s_1; s_2) + U_2(s_2; s_1)] - \bar{u}_2 \\ &= EV - E[y_1 + y_2] - \bar{u}_2 \end{aligned}$$

Because $EV$ is 1, $E[y_1 + y_2] \leq \frac{1}{2} + E[f_1^{-1}(s_2)] - \bar{u}_2$. If the full surplus is extracted, $E[f_1^{-1}(s_2)] = \frac{1}{2}$.

Suppose, by way of contradiction, Worker 2 wins with probability less than or equal to one half. Then, $E[f_1^{-1}(s_2)] < E[f_2^{-1}(s_2)] \leq \frac{1}{2}$, which is a contradiction. □

This exemplifies what we mean by *covert discrimination.* We have covert discrimination when the weaker player, Worker 2, is receives a larger expected prize in equilibrium. No typical contest has this property. If the costs are scaled, covert discrimination is not possible.

**Theorem 1.** *If* $f_2(x) = \alpha f_1(x)$ *for some* $\alpha \in (0,1)$*, the optimal contest success function is an all-pay auction with a bid cap at* $\bar{s} \equiv f_2(1/2)$*. That is,*

$$p(s_i; s_{-i}) \equiv \begin{cases} 0 & \text{if } s_i < s_{-i} \leq \bar{s} \text{ or } s_i > \bar{s} \\ \frac{1}{2} & \text{if } s_i = s_{-i} \\ 1 & \text{if } s_i < s_{-i} \leq \bar{s} \text{ or } s_{-i} > \bar{s}. \end{cases}$$

*The output from this optimal contest is* $y_1^* + y_2^* = \frac{1+\alpha}{2}$*.*

*Proof.* Combine the inequality from Lemma 1, $E[y_1 + y_2] \leq \frac{1}{2} + E[f_1^{-1}(s_2)] - \bar{u}_2$, with our assumption. This gives,

$$E[y_1 + y_2] \leq \frac{1}{2} + \alpha E[f_2^{-1}(s_2)] - \bar{u}_2.$$

We want to show that $E[f_2^{-1}(s_2)] \leq 0.5$ (i.e., that covert discrimination is impossible). To see this, note that because the players can switch to each other's strategies, Worker 2 must prefer his strategy to copying Worker 1:

$$\frac{1}{2} - E[f_2^{-1}(s_1)] \leq E[p(s_2; s_1)] - E[f_2^{-1}(s_2)],$$

and Worker 1 must prefer his strategy to copying Worker 2:

$$\frac{1}{2} - \alpha E[f_2^{-1}(s_2)] \leq E[p(s_1; s_2)] - \alpha E[f_2^{-1}(s_1)].$$

Combining the two and substituting $E[p(s_1; s_2)] + E[p(s_2; s_1)] = 1$ yields,

$$\alpha^{-1}\left(E[p(s_2; s_1)] - \frac{1}{2}\right) \leq E[p(s_2; s_1)] - \frac{1}{2},$$

which implies $E[f_2^{-1}(s_2)] \leq 0.5$. Therefore, $E[y_1 + y_2] \leq \frac{1+\alpha}{2}$, a bound attained by the proposed contest. □

Note that the nonexistence of covert discrimination in this setting is used in the proof. In general, covert discrimination is often possible.

**Example 1** (Full surplus extraction)**.** Suppose that two workers have self-promotion functions that we define by inverse as

$$f_1^{-1}(x) \equiv \begin{cases} \frac{2x}{3} & \text{if } x \leq 0.5 \\ \frac{1}{3} + (x - 0.5) & \text{if } x > 0.5 \end{cases}$$

$$f_2^{-1}(x) \equiv \begin{cases} x & \text{if } x \leq 0.5 \\ \frac{1}{2} + \frac{2}{3}(x - 0.5) & \text{if } x \in (0.5, 0.75] \\ \frac{2}{3} + (x - 0.75) & \text{if } x > 0.75 \end{cases}$$

Consider the following contest success function,

$$p(s_i; s_{-i}) \equiv \begin{cases} 0 & \text{if } s_i < s_{-i} \text{ and } s_i < 0.5 \\ \frac{1}{2} - \frac{2}{3}(s_{-i} - s_i) & \text{if } 0.5 \leq s_i < s_{-i} \text{ and } 0.5 < s_{-i} \\ \frac{1}{2} & \text{if } s_i = s_{-i} \\ \frac{1}{2} + \frac{2}{3}(s_i - s_{-i}) & \text{if } 0.5 \leq s_{-i} < s_i \text{ and } 0.5 < s_i \\ 1 & \text{if } s_{-i} < s_i \text{ and } s_{-i} < 0.5, \end{cases}$$

where the second and fourth lines are capped at zero and one. This contest is an all-pay auction until both players play at least 0.5 and a difference-form contest thereafter. There is an equilibrium where Worker 1 plays $s_1^* = \frac{1}{2}$ and Worker 2 plays $s_2^* = \frac{3}{4}$. The real outputs are $y_1^* = f_1^{-1}(0.5) = \frac{1}{3}$ and $y_1^* = f_2^{-1}(0.75) = \frac{2}{3}$. There is full surplus extraction in this game.

To see why there is no strategic deviations, consider the endogenous contest success functions for each player. Worker 1 faces the following contest success function in equilibrium:

$$p(s_1; 0.75) \equiv \begin{cases} 0 & \text{if } s_1 < 0.5 \\ \frac{1}{2} - \frac{2}{3}(0.75 - s_1) & \text{if } s_1 \in [0.5, 0.75) \\ \frac{1}{2} & \text{if } s_1 = 0.75 \\ \frac{1}{2} + \frac{2}{3}(s_1 - 0.75) & \text{if } s_1 > 0.75. \end{cases}$$

The slope is not large enough for Worker 1 to want to increase past $s_1^* = \frac{1}{2}$. Clearly, no decrease in the score will increase the payoff either. Worker 2 faces the following contest success function in equilibrium:

$$p(s_2; 0.5) \equiv \begin{cases} 0 & \text{if } s_2 < 0.5 \\ \frac{1}{2} & \text{if } s_2 = 0.5 \\ \frac{1}{2} + \frac{2}{3}(s_2 - 0.5) & \text{if } s_2 > 0.5. \end{cases}$$

Worker 2 is indifferent between all scores in $[0.5, 0.75]$. The contest success function is exactly equal to his costs on this interval. △

Lemma 1 showed that covert discrimination is required to extract the full surplus because the self-promotion abilities are ranked. In Example 1, the manager uses the fact that the *marginal* self-promotion abilities need not be ranked to extract the full surplus.

The combined all-pay auction with difference form contest used in Example 1 is a very useful tool for covert discrimination. However, it is not a solution for all cases. In fact, unranked marginal costs are not sufficient to ensure that full surplus extraction is possible. This fact is made clear by the following proposition.

**Proposition 1.** *If $f_1(x) \geq 2f_2(x)$ for all $x \in [0, 1]$, full surplus extraction is not possible.*

*Proof.* By way of contradiction, suppose the full surplus is extracted, then both players have a payoff of zero. If Worker 2 has a payoff of zero,

$$E[p(s_1; s_2)] - E[f_2^{-1}(s_2)] = 0.$$

If Worker 1 imitates this strategy, he receives

$$0 \geq 0.5 - E[f_1^{-1}(s_2)] \geq 0.5 - 0.5E[f_2^{-1}(s_2)].$$

This implies the following contradiction

$$0 \geq 1 - E[f_2^{-1}(s_2)] > E[p(s_1; s_2)] - E[f_2^{-1}(s_2)] = 0.$$

Here we use the fact that Worker 2 cannot win with probability 1. This is because this would imply that Worker 1 plays zero, which Worker 2 would imitate for a positive payoff. □

Proposition 1 demonstrates that the absolute scale of the functions is important. Full surplus extraction is not possible with a fixed prize in all cases where marginal costs are not ranked. However, if the manager is able to adjust the prize endogenously, full surplus extraction is always possible.

## 3.3 Endogenous prize

When the designer can construct a prize with spillovers, full surplus extraction is possible.

**Theorem 2** (Full surplus extraction)**.** *If the marginal inverse self-promotion technologies, are not ranked such that* $(f_2^{-1})'(t) < (f_1^{-1})'(t)$ *for some* $t$ *such that* $(f_1^{-1})'(t) + (f_2^{-1})'(t) < EV$*, there exists an optimal contest of the form* $v(s; y) = \check{v}^\star(y)$ *with the auction contest success function,*

$$p(s_i; s_{-i}) = \begin{cases} 1 & \text{if } s_i > s_{-i} \\ 0.5 & \text{if } s_i = s_{-i} \\ 0 & \text{if } s_i < s_{-i}, \end{cases}$$

*such that surplus is fully extracted.*

Theorem 2 shows that it is possible for the principal to achieve the first-best outcome using an all-pay auction with spillovers. The prize does not even have to depend on the player's own score. The optimal prize is easily constructed. The function is a continuous approximation of a step function that is larger than $\bar{v}$ on the region where $(f_2^{-1})'(t) > (f_1^{-1})'(t)$, but is smaller in the region where $(f_2^{-1})'(t) < (f_1^{-1})'(t)$.

**Example 2.** Worker 1 self-promotes with inverse function $f_1^{-1}(s) = s^2$, whereas Player 2 self-promotes according to $f_2^{-1}(s) = s$.

For convenience, we are going to use a prize budget of 0.9 instead of 1. That is, $EV \leq 0.9$. Note that on the interval $(0, 0.9]$, $f_2^{-1}(s) > f_1^{-1}(s)$ but marginal costs are not similarly ranked. Because of this, there exists a prize such that neither player has a mass point at zero.

The fixed prize that maximizes effort, given the designer's constraint, is $v = 0.9$.

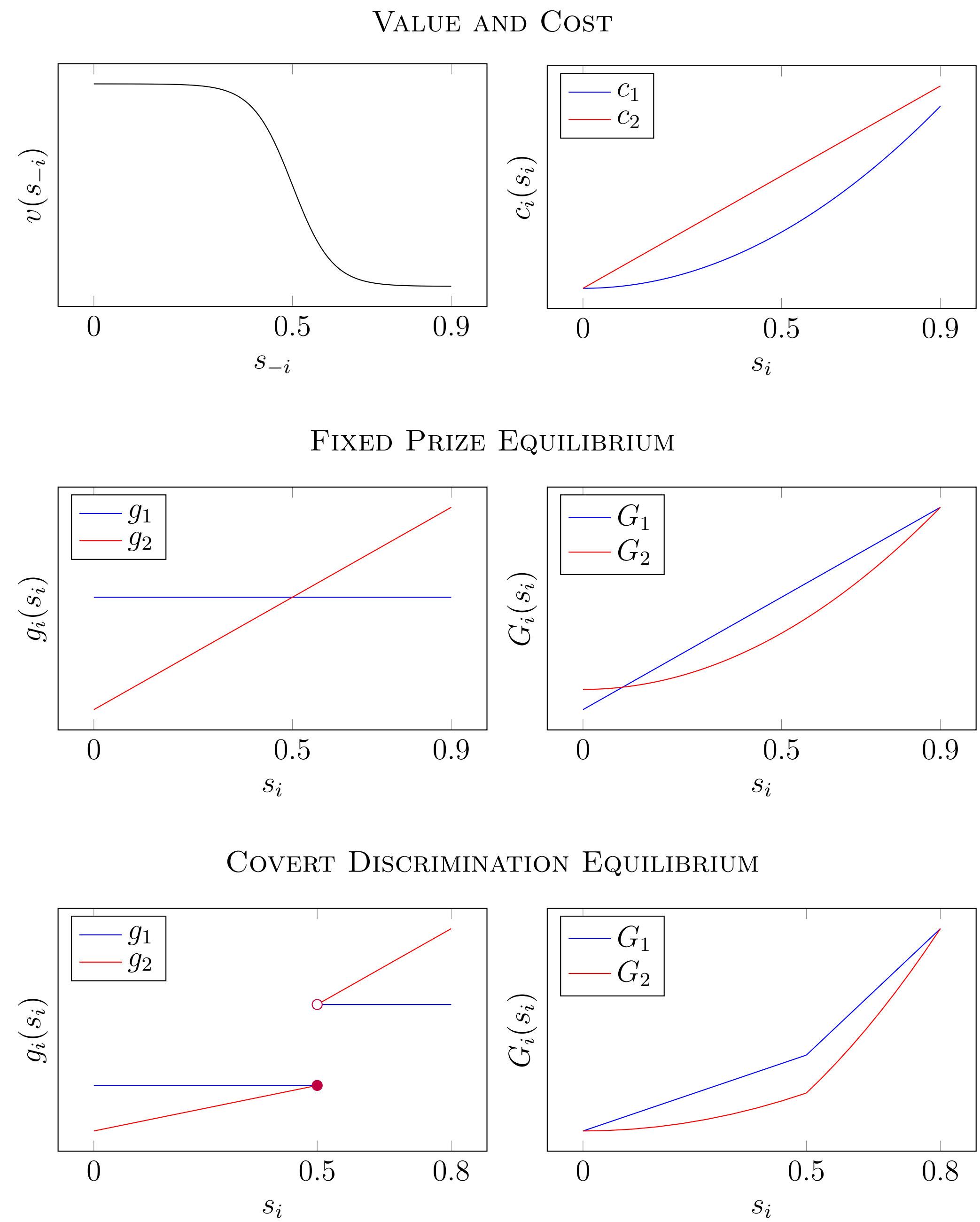


Figure 1: Plots of equilibrium densities (left), distributions (right), and primitives (top) from Example 2. In both cases, Player 2 places more density on scores in $(0.5, 0.9]$ than Player 1. Because the prize is sharply less valuable when your opponent plays in this interval, in equilibrium, the prize is not as valuable for Player 1. This drives the increase in the contests' competitiveness.

In this case, the players' equilibrium strategies' cumulative distribution functions are

$$G_1(s) = \frac{10s}{9} \qquad G_2(s) = \frac{1}{10} + \frac{10s^2}{9}.$$

The expected output produced by both competitors is $0.9 - (0.9 - (0.9)^2) = 0.81$. Now, consider the following prize:

$$\check{v}(x) = \begin{cases} \frac{4}{3} & \text{if } x \leq \frac{1}{2} \\ \frac{11}{25} & \text{if } x > \frac{1}{2} \end{cases}$$

where $x$ is the opponent's output. That is, the value of the prize to one player depends explicitly on the output produced by their opponent. With this particular reward choice, $EV = 0.9$ and total output is 0.9.

This example is visualized in Figure 1, and demonstrates that covert discrimination can be revenue improving. The key to this example lies in how the stronger player 1's marginal costs are above player 2's for $s > 1/2$, even if the costs themselves aren't. By carefully crafting the prize values, the designer is able to exploit the marginal disadvantage of the stronger player to improve the contests' competitiveness without having to rely on differential treatment of the participants. △

# References


Betto, Maria and Matthew Wildrick Thomas (2021). "Asymmetric All-Pay Contests with Spillovers". en. In: *Working Paper*.

Brown, Jennifer (2011). "Quitters never win: The (adverse) incentive effects of competing with superstars". In: *Journal of Political Economy* 119, pp. 982–1013.

Che, Yeon-Koo and Ian Gale (2003). "Optimal design of research contests". In: *American Economic Review* 93, pp. 646–671.

Epstein, Gil S, Yosef Mealem, and Shmuel Nitzan (2013). "Lotteries vs. All-Pay Auctions in Fair and Biased Contests". In: *Economics & Politics* 25, pp. 48–60.

Franke, Jörg et al. (2013). "Effort maximization in asymmetric contest games with heterogeneous contestants". In: *Economic Theory* 52, pp. 589–630.

Holzer, Harry and David Neumark (2000). "Assessing affirmative action". In: *Journal of Economic literature* 38, pp. 483–568.

Jönsson, Stefan and Armin Schmutzler (2013). “All-pay auctions: Implementation and optimality”. In: *Working Paper*.

Mealem, Yosef and Shmuel Nitzan (2014). “Equity and effectiveness of optimal taxation in contests under an all-pay auction”. In: *Social Choice and Welfare* 42, pp. 437–464.

Nti, Kofi O (1999). “Rent-seeking with asymmetric valuations”. In: *Public Choice* 98, pp. 415–430.

Roemer, John E (1998). *Equality of opportunity*. Harvard University Press.

# 4 Appendix

## 4.1 Proof of Theorem 2

Define the sets $L, H$ as $L \equiv \{x \in [0, \bar{v}] | c_1'(x) > c_2'(x)\}$ and $H \equiv [0, \infty) - L = \{x \in [0, \infty) | c_1'(x) \leq c_2'(x)\}$. We propose the following step contest and show that $R = \bar{v}$

$$\check{v}^\star(x) = \begin{cases} v_l & \text{if } x \in L \\ v_h & \text{if } x \in H \end{cases}$$

for some $v_h \geq v_l > 0$. That is, we need to show that there exist $v_h \geq v_l > 0$ such that

$$R \equiv \int_0^{\bar{s}} c_1(x)\, dG_1(x) + \int_0^{\bar{s}} c_2(x)\, dG_2(x) = \bar{v} \tag{2}$$

where $\bar{s}, G_1, G_2$ are equilibrium objects. When there is no mass point in either player's strategy distribution,

$$g_i(x) = \frac{c'_{-i}(x)}{\check{v}^\star(x)} = \begin{cases} \frac{c'_{-i}(x)}{v_l} & \text{if } x \in L \\ \frac{c'_{-i}(x)}{v_h} & \text{if } x \in H. \end{cases}$$

There are no mass points if and only if $\int_0^{\bar{s}} g_1(x)\,dx = \int_0^{\bar{s}} g_2(x)\,dx = 1$. In this case, this is equivalent to

$$\begin{aligned} G_{-i}(\bar{s}) &= \int_0^{\bar{s}} g_{-i}(x)\,dx \\ &= \int_0^{\bar{s}} \frac{c_i'(x)}{\check{v}^\star(x)}\,dx \\ &= \frac{1}{v_l}\int_{L\cap S} c_i'(x)\,dx + \frac{1}{v_h}\int_{H\cap S} c_i'(x)\,dx \\ &= \frac{1}{v_l}c_i(\bar{s}) - \left(\frac{1}{v_l} - \frac{1}{v_h}\right)\int_{H\cap S} c_i'(x)\,dx = 1 \end{aligned} \tag{3}$$

for both $i$.

In this case, equation (2) can be rewritten as

$$\begin{aligned} R &\equiv \int_0^{\bar{s}} c_1(x)\,dG_1(x) + \int_0^{\bar{s}} c_2(x)\,dG_2(x) \\ &= \int_0^{\bar{s}} \frac{1}{\check{v}^\star(x)}\left(c_1(x)c_2'(x) + c_1'(x)c_2(x)\right)\,dx \\ &= \frac{1}{v_l}\int_{L\cap S} \frac{dc_1(x)c_2(x)}{dx}\,dx + \frac{1}{v_h}\int_{H\cap S} \frac{dc_1(x)c_2(x)}{dx}\,dx \\ &= \frac{1}{v_l}c_1(\bar{s})c_2(\bar{s}) - \left(\frac{1}{v_l} - \frac{1}{v_h}\right)\int_{H\cap S} \frac{dc_1(x)c_2(x)}{dx}\,dx = \bar{v}. \end{aligned} \tag{4}$$

If we substitute $l \equiv \frac{1}{v_l}$ and $h \equiv \frac{1}{v_l} - \frac{1}{v_h}$ into (3) and (2), we get the following system of equations

$$\begin{aligned} \bar{v} &= lc_1(\bar{s})c_2(\bar{s}) - h\int_{H\cap S} \frac{dc_1(x)c_2(x)}{dx}\,dx \\ 1 &= lc_1(\bar{s}) - h\int_{H\cap S} c_1'(x)\,dx \\ 1 &= lc_2(\bar{s}) - h\int_{H\cap S} c_2'(x)\,dx \end{aligned}$$

We can isolate $l, h$ from the second and third equation

$$l = \frac{\int_{H\cap S} c_2'(x) - c_1'(x)\,dx}{D(\bar{s})} > \frac{c_2(\bar{s}) - c_1(\bar{s})}{D(\bar{s})} = h$$

where

$$D(\bar{s}) \equiv c_2(\bar{s})\int_{L\cap S} c_1'(x)\,dx - c_1(\bar{s})\int_{L\cap S} c_2'(x)\,dx > 0.$$

If an $\bar{s}$ exists, it is obvious that $\bar{s} > k \equiv \inf L$. Otherwise, none of these equations are defined. So, the only thing to confirm is that there exists a $\bar{s} > k$ such that

$$Q(\bar{s}) \equiv \frac{c_1(\bar{s})c_2(\bar{s}) \int_{H\cap S} c_2'(x) - c_1'(x)\, dx - (c_2(\bar{s}) - c_1(\bar{s})) \int_{H\cap S} \frac{dc_1(x)c_2(x)}{dx}\, dx}{c_1(\bar{s}) \int_{H\cap S} c_2'(x)\, dx - c_2(\bar{s}) \int_{H\cap S} c_1'(x)\, dx} = \bar{v}$$

We know that $c_1'(k) = c_2'(k)$.

$$\lim_{x\to k} Q(x) = \lim_{x\to k} \frac{c_1(x)c_2(x) \int_{H\cap S(x)} c_2'(y) - c_1'(y)\, dy - (c_2(x) - c_1(x)) \int_{H\cap S(x)} \frac{dc_1(y)c_2(y)}{dy}\, dy}{c_1(x) \int_{H\cap S(x)} c_2'(y)\, dy - c_2(x) \int_{H\cap S(x)} c_1'(y)\, dy}$$

By L'Hopital's rule

$$\begin{aligned}
&= \frac{\frac{dc_1(k)c_2(k)}{dk}(c_2(k) - c_1(k)) - (c_2'(k) - c_1'(k))\, c_1(k)c_2(k)}{c_1'(k)c_2(k) - c_1(k)c'2(k)} \\
&= \left(\frac{c_1'(k)c_2(k) + c_1(k)c_2'(k)}{c_1'(k)c_2(k) - c_1(k)c_2'(k)}\right)(c_2(k) - c_1(k)) \\
&= c_1(k) + c_2(k)
\end{aligned}$$

Additionally,

$$\begin{aligned}
Q(x) &= \frac{c_1(x)c_2(x) \int_{H\cap S(x)} c_2'(y) - c_1'(y)\, dy - (c_2(x) - c_1(x)) \int_{H\cap S(x)} \frac{dc_1(y)c_2(y)}{dy}\, dy}{c_1(x) \int_{H\cap S(x)} c_2'(y)\, dy - c_2(x) \int_{H\cap S(x)} c_1'(y)\, dy} \\
&> \frac{c_1(x)c_2(x) \int_{H\cap S(x)} c_2'(y) - c_1'(y)\, dy - (c_2(x) - c_1(x))\, c_1(x)c_2(x)}{c_1(x) \int_{H\cap S(x)} c_2'(y)\, dy - c_2(x) \int_{H\cap S(x)} c_1'(y)\, dy} \\
&= \frac{c_1(x)c_2(x) \int_{L\cap S(x)} c_1'(y) - c_2'(y)\, dy}{c_1(x) \int_{H\cap S(x)} c_2'(y)\, dy - c_2(x) \int_{H\cap S(x)} c_1'(y)\, dy} \\
&> \frac{c_1(x)c_2(x) \int_{L\cap S(x)} c_1'(y) - c_2'(y)\, dy}{c_1(x)c_2(x) - c_2(x) \int_{H\cap S(x)} c_1'(y)\, dy} \\
&= \frac{c_1(x) \int_{L\cap S(x)} c_1'(y) - c_2'(y)\, dy}{\int_{L\cap S(x)} c_1'(y)\, dy} \\
&= c_1(x)\left(1 - \frac{\int_{L\cap S(x)} c_2'(y)\, dy}{\int_{L\cap S(x)} c_1'(y)\, dy}\right)
\end{aligned}$$

So, $\lim x \to \infty Q(x) = \infty$. By continuity, there exists a $\bar{s}$ such that $Q(\bar{s}) = \bar{v}$.