Choice over Assessments

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Section 1

Introduction

Selecting assessments

- **Assessment** is lottery over *scores* which depends on agent's type
- Scores reveal information about agent's type
- Agent choose assessment to increase expected score (e.g., SAT vs ACT)

This is not choice *under* uncertainty. *It is choice **of** uncertainty.*

Assortative matching intuition

Intuitively, higher types prefer more accurate assessments:

- Lowest type wants assessment that reveals no information
- Highest type prefers perfectly revealing assessment

Want to formalize and study this intuition for comparing assessments.

Roadmap

- Model
- Assortative matching result
- Relationship to other orders
- Menu design and applications
- Extensions and repeated testing

Section 2

Model

Model

- Agents have private types $\theta \in \Theta$ distributed by G
- Scores, $s \in S$, distributed by assessments, F_i, conditional on type
- Agent's utility over scores, u, weakly increasing
- Agent payoff is $U(i, \theta) = \int_S u(s) dF_i(s|\theta)$ from choosing assessment F_i
- $\mathcal{I}_\theta := \arg\max_{\hat{i}} U_{\hat{i} \in \mathcal{I}}(s, \theta)$ denotes the set of assessments that type θ prefers

Definition of types/assessments

Higher types FOSD lower types' distributions for each assessment

Assumption (type order)

For all assessments, F_i, $s \in S$ and all $\theta, \theta' \in \Theta$ with $\theta < \theta'$,

$$F_i(s|\theta') \leq F_i(s|\theta)$$

Decreasing differences property

Definition (decreasing differences)

Assessments satisfy DD (submodularity) iff for all $s \in S$, $i, j \in \mathcal{I}$ with $i < j$ and $\theta, \theta' \in \Theta$ with $\theta < \theta'$,

$$F_j(s|\theta') - F_i(s|\theta') \leq F_j(s|\theta) - F_i(s|\theta)$$

We will see DD is sufficient for *weak* assortative matching

Section 3

Assortative matching result

Basic MCS Results

Theorem

DD holds if and only if the expected utility

$$U(i,\theta) = \int_{s\in S} u(s)dF_i(s|\theta)$$

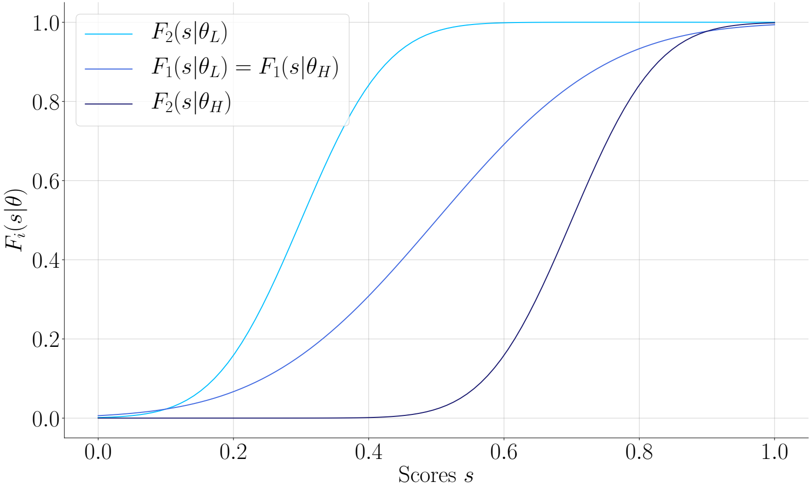
is supermodular for any monotone utility function.

Sufficiency Proof Necessity Proof

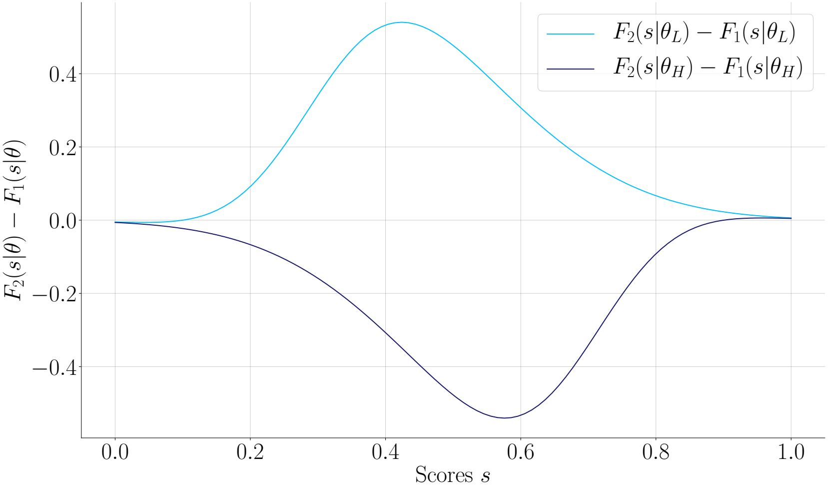
Corollary

DD implies $\mathcal{I}_{\theta'}$ *strong-set order dominates* $\mathcal{I}_\theta$ *for all* $\theta' > \theta$.

Example: Normal Distributions



Example: Normal Distributions



Example I

Suppose F_2 reveals the agent's type with certainty while F_1 is uniform independently of type. For any $\theta < \theta'$,

$$F_2(s|\theta') - F_1(s|\theta') = \mathbf{1}_{\{s \geq \theta'\}} - s \leq \mathbf{1}_{\{s \geq \theta\}} - s = F_2(s|\theta) - F_1(s|\theta)$$

Example II

Assume a family $\{F_\alpha(\cdot|\theta) : \alpha \in [0,1]\}$ of cdfs of distributions that, with probability α, perfectly reveals the agent's type and, with probability $1-\alpha$, draws a random score from the $\mathcal{U}[0,1]$ distribution. Then,

$$F_\alpha(s|\theta) = \mathbf{1}_{\{s \geq \theta\}}\alpha + s(1-\alpha)$$

Now fix $\alpha' > \alpha$ and $\theta' > \theta$. Then,

$$\begin{aligned} F_{\alpha'}(s|\theta') - F_\alpha(s|\theta') &= \Big(\mathbf{1}_{\{s \geq \theta'\}} - s\Big)(\alpha' - \alpha) \\ &\leq \Big(\mathbf{1}_{\{s \geq \theta\}} - s\Big)(\alpha' - \alpha) \\ &= F_{\alpha'}(s|\theta) - F_\alpha(s|\theta). \end{aligned}$$

In this case, a higher assessment corresponds to a higher α. Here, our ordering coincides with Blackwell informativeness. We will see later that this is not always the case.

Section 4

Relationship to other orders

Relationship with Blackwell (2 scores)

Lemma

If $S := \{s_L, s_H\}$, the Blackwell informativeness criterion implies DD.

Proof.

Suppose assessment i is a garbling of assessment j:

$$F_j(s_L|\theta') - F_i(s_L|\theta') = p_j(s_L|\theta')(1 - z(s_L, s_L)) - z(s_L, s_H)p_j(s_H|\theta')$$

$$\leq p_j(s_L|\theta)(1 - z(s_L, s_L)) - z(s_L, s_H)p_j(s_H|\theta) = F_j(s_L|\theta) - F_i(s_L|\theta).$$

■

Relationship with Blackwell (2 scores)

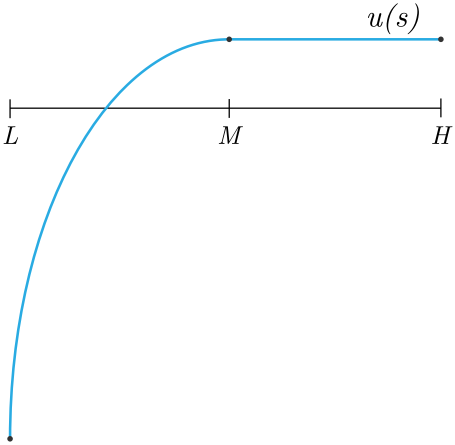
Blackwell is sufficient for DD, but not necessary. Consider P_i and P_j s.t.

$$p_i(s_L|\theta) = 1 - \epsilon \qquad p_i(s_L|\theta') = \frac{1}{2}$$
$$p_j(s_L|\theta) = \frac{1}{2} \qquad p_j(s_L|\theta') = 0$$

assessment i is not a garbling of j for $\epsilon < \frac{1}{4}$. Yet, DD is satisfied:

$$\underbrace{F_j(s_L|\theta) - F_j(s_L|\theta')}_{\frac{1}{2}} \geq \underbrace{F_i(s_L|\theta) - F_i(s_L|\theta')}_{\frac{1}{2}-\epsilon}$$

Blackwell does not imply DD with 3 or more scores



In general, Blackwell does not imply DD

Intuitively, a medium type may care more about accuracy than a high type if the difference in utility from a medium and low score is sufficiently large. Counterexample

Relationship with concordance ordering

Definition (Concordance ordering)
assessment j dominates i in the concordance ordering iff $F_j(s) = F_i(s)$ and

$$p_j(S \leq s, \Theta \leq \theta) \geq p_i(S \leq s, \Theta \leq \theta)$$

If the marginals are the same ($F_j(s) = F_i(s)$) DD implies the concordance ordering. The converse is true if there are only two scores. Proof

Relationship with concordance ordering

Because the underlying distribution of types does not depend on the assessment chosen, we can divide both sides to get a definition in terms of conditionals:

$$F_j(s|\Theta \leq \theta) \geq F_i(s|\Theta \leq \theta)$$

Because our problem is two dimensional, the concordance ordering is equivalent to greater weak association, the supermodular ordering, the convex-modular ordering, and the dispersion ordering.

Section 5

Menu design and applications

Collecting information

If we do not use the information, we can collect types:

- Construct a menu of garblings in the DD order
- Obtain types from observing the choice of assessment

However, this does not allow use of types in a way that affects agents.

Menu design motivation

Can we design assessment menus to make scores more accurate?

Sort of.

- Use assortative matching to reveal information
- Need additional assumptions to misalign preferences of principal/agent

Simplest example

Professor is writing graduate admissions letters for undergrads

- Has assessment with three scores: $1, 2, 3$
- Students have two types: θ_L, θ_H
- Assume student utility, u, is concave
- Professor wants to write letters for θ_H only
- Assessment usually assigns θ_L to 1, but sometimes assigns 2 or 3

With this assessment, professor must occasionally be writing letters for θ_L.

Simplest example

Professor offers a menu of assessment and garbling that only gives score 2

- Students with θ_L will take the garbling
- Any student with score 3 must have type θ_H
- Professor can write letters for θ_H only

Note: We used concavity of u to ensure that students do not also only care about score 3. If they did, any menu would be detrimental.

Section 6

Extensions and repeated testing

Choice of assessments under repetition

Suppose the agent may retake assessments at cost c

- **New question: How does her choice of assessment change?**
- This is now an *optimal stopping/search problem*.

Consider type θ. Suppose she chooses assessment i because she finds it preferable to any other assessment. Assume she has a current best score of $s^\star$ and is considering whether to stop.

Assume each trial costs c, and that $U(i, \theta) - c > u(\underline{s})$ for all $i \in \mathcal{I}$ and all $\theta \in \Theta$.

If continuing is preferable, then the value of doing so is

$$V_i(s^\star,\theta) = (1 - F_i(s^\star|\theta))E[\max\{u(s), V_i(s,\theta)\}|s > s^\star] + F_i(s^\star|\theta)V_i(s^\star,\theta) - c$$

$$\implies V_i(s^\star,\theta) = E[\max\{u(s), V_i(s,\theta)\}|s > s^\star] - \frac{c}{(1 - F_i(s^\star|\theta))}$$

The value of stopping is simply $u(s^\star)$. Thus, type θ stops at $s^\star$ if and only if

$$E[u(s)|s > s^\star, \theta, i] - \frac{c}{(1 - F_i(s^\star|\theta))} \leq u(s^\star)$$

$$\implies \frac{\int_{s>s^\star} u(s)dF_i(s|\theta) - c}{(1 - F_i(s^\star|\theta))} \leq u(s^\star)$$

We let $s^\star_{\theta i} := \arg\max_{s^\star \in S}\left\{\frac{\int_{s>s^\star} u(s)dF_i(s|\theta)-c}{(1-F_i(s^\star|\theta))} \leq u(s^\star)\right\}$ denote the set of optimal stopping scores for type θ at assessment i. Note that $\theta' > \theta \iff s^\star_{\theta' i} \geq_{SSO} s^\star_{\theta i}$.

Let:

$$U^\star(i,\theta) := \int_{s\in S} u(s)dF_i(s|\theta, s > s^\star_{\theta i}) - \frac{c}{(1 - F_i(s^\star|\theta))}$$

It is necessary and sufficient for the supermodularity of $U^\star$ that, for $j > i$ and $s \geq \max_{\tilde{\theta},k}\{s^\star_{\tilde{\theta}k}\}$,

$$F_j(s|\theta', s > s^\star_{\theta' j}) - F_i(s|\theta', s > s^\star_{\theta' i}) \leq F_j(s|\theta, s > s^\star_{\theta j}) - F_i(s|\theta, s > s^\star_{\theta i})$$

since the total expected costs are decreasing in type.

Example: repeated assessments with low costs

Suppose that c is low enough that all players choose a $\bar{s}$ as their cutoff
Then, weak assortative matching is equivalent to

$$\frac{p_i(\bar{s}|\theta_L) - p_j(\bar{s}|\theta_L)}{p_i(\bar{s}|\theta_L)p_j(\bar{s}|\theta_L)} \geq \frac{p_i(\bar{s}|\theta_M) - p_j(\bar{s}|\theta_M)}{p_i(\bar{s}|\theta_M)p_j(\bar{s}|\theta_M)} \geq \frac{p_i(\bar{s}|\theta_H) - p_j(\bar{s}|\theta_H)}{p_i(\bar{s}|\theta_H)p_j(\bar{s}|\theta_H)}$$

Because of the type definition, this is implied by

$$p_i(\bar{s}|\theta_L) - p_j(\bar{s}|\theta_L) \geq p_i(\bar{s}|\theta_M) - p_j(\bar{s}|\theta_M) \geq p_i(\bar{s}|\theta_H) - p_j(\bar{s}|\theta_H)$$

which is implied by DD.

Thank You!

Section 7

Proofs

Sufficiency of DD

Proof.

Assume $j \in \mathcal{I}_\theta$ and let $i < j$. If $i \in \mathcal{I}_{\theta'}$, then, using integration by parts,

$$
\begin{aligned}
0 &\leq \int_{s\in S} u(s)dF_i(s|\theta') - \int_{s\in S} u(s)dF_j(s|\theta') \\
&= \left(u(\bar{s}) - \int_{s\in S} F_i(s|\theta')du(s)\right) - \left(u(\bar{s}) - \int_{s\in S} F_j(s|\theta')du(s)\right) \\
&= \int_{s\in S}(F_j(s|\theta') - F_i(s|\theta'))du(s) \\
&\leq \int_{s\in S}(F_j(s|\theta) - F_i(s|\theta))du(s) \\
&= \int_{s\in S} u(s)dF_i(s|\theta) - \int_{s\in S} u(s)dF_j(s|\theta)
\end{aligned}
$$

Since θ prefers j, the above implies that θ must also prefer i, i.e, $i \in \mathcal{I}_\theta$. ∎

Necessity of DD

Proof.

Suppose, by means of contradiction, that DD is violated. That is, there exists $s^\star$ such that

$$F_j(s^\star|\theta') - F_i(s^\star|\theta') > F_j(s^\star|\theta) - F_i(s^\star|\theta) \tag{1}$$

Consider the following weakly monotone utility function:

$$u(s) = \begin{cases} 0 & \text{if } s < s^\star \\ 1 & \text{if } s \geq s^\star \end{cases}$$

Then the expected utility from assessment k for type θ is $1 - F_k(s^\star|\theta)$. By (1) SM of the expected utility is violated because:

$$EU_j(\theta') - EU_i(\theta') < EU_j(\theta) - EU_i(\theta)$$

Blackwell counterexample

With three scores, Blackwell does not imply DD. To see why, consider $S := \{s_L, s_M, s_H\}$, $\Theta = \{\theta_M, \theta_H\}$ and $u(s_L) < u(s_M) = u(s_H)$. Let assessment j be perfectly revealing, i.e., $p_j(s_M|\theta_M) = p_j(s_H|\theta_H) = 1$ and let assessment i be a garbling of j where

$$p_i(s_L|\theta_M) = p_i(s_M|\theta_L) = p_i(s_M|\theta_H) = p_i(s_H|\theta_H) = \frac{1}{2}$$

Then, type θ_M really wants to avoid getting s_L, whereas type θ_H doesn't have to worry about it since it has no chance of obtaining it. Note that the example above violates the condition in DD:

$$F_j(s_L|\theta_M) - F_i(s_L|\theta_M) = -\frac{1}{2} < 0 = F_j(s_L|\theta_H) - F_i(s_L|\theta_H)$$

Sufficiency of concordance ordering

Proof.

$$
\begin{aligned}
&E_\theta\Big[F_j(s|\tilde{\theta}) - F_i(s|\tilde{\theta})|\tilde{\theta} \le \theta\Big] \Pr\Big(\tilde{\theta} \le \theta\Big) \\
&\quad + E_\theta\Big[F_j(s|\tilde{\theta}) - F_i(s|\tilde{\theta})|\tilde{\theta} > \theta\Big] \Pr\Big(\tilde{\theta} > \theta\Big) = 0 \qquad (2)\\
&\quad \Longrightarrow \; E_\theta\Big[F_j(s|\tilde{\theta}) - F_i(s|\tilde{\theta})|\tilde{\theta} > \theta\Big] \le 0 \qquad (3)\\
&\quad \Longrightarrow \; \int_{\theta \in \Theta} \Big(F_j(s|\tilde{\theta}) - F_i(s|\tilde{\theta})\Big) dF(\tilde{\theta}|\tilde{\theta} > \theta) \le 0 \\
&\quad \Longrightarrow \; F_j(s|\tilde{\theta} > \theta) - F_i(s|\tilde{\theta} > \theta) \le 0
\end{aligned}
$$

Where we used $F_i(s) = F_j(s)$ in line (2) and Definition 1 to derive (3). ■